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Instability of localized buckling modes in a one-dimensional strut model

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Stability of localized solutions arising in a fourth-order differential equation modelling struts is investigated. It was shown by Buffoni *et al.* in 1996 that the model exhibits many multimodal buckling states bifurcating from a primary buckling mode. In this article, using analytical and numerical techniques, the primary mode is shown to be unstable under dead loading for all axial loads, while it is likely to be stable under rigid loading for small axial loads. Furthermore, for general reversible or conservative systems, stability of the multimodal solutions is established assuming stability of the primary state. Since this hypothesis is not satisfied for the buckling mode arising in the strut model, any multimodal buckling state will be unstable under dead and rigid loading.

1. Introduction

In this article, localized solutions of the fourth-order ordinary differential equation

$$u_{xxxx} + Pu_{xx} + u - u^2 = 0, \quad x \in \mathbb{R}, \tag{1.1}$$

are investigated. Equation (1.1) describes equilibrium states of a strut resting on an elastic foundation with a nonlinear softening restoring force (see, for example, Hunt *et al.* 1989). Here, x and u are the arc length along the strut and vertical displacement, respectively. The parameter P denotes the axial load, while the bending stiffness has been rescaled to unity. The underlying geometry is illustrated in figure 1.

Note that (1.1) is Hamiltonian with energy given by

$$H(u) = u_x u_{xxx} - \tfrac{1}{2}u_{xx}^2 + \tfrac{1}{2}Pu_x^2 + \tfrac{1}{2}u^2 - \tfrac{1}{3}u^3.$$

By definition, *localized* solutions h of (1.1) satisfy

$$\lim_{x\to\pm\infty} h(x) = 0,$$

i.e. they correspond to homoclinic solutions of (1.1). It was shown by Amick & Toland (1992) that (1.1) has an even homoclinic solution h for each $P \in (-\infty, -2+\eta)$ for some $\eta > 0$. In addition, they proved uniqueness for $P \leqslant -2$, whence we refer to these localized solutions as the *primary* buckling modes. The solutions h are *transversely constructed*, i.e. stable and unstable manifolds of the zero equilibrium of (1.1) intersect transversely at $u = h(0)$ in the zero level set of the energy H (see Buffoni *et al.* 1996). Using the results of Devaney (1976), Buffoni *et al.* (1996) also proved that for any $P \in (-2, -2+\eta)$, with $\eta > 0$ sufficiently small, infinitely many buckling modes bifurcate from the primary state. The bifurcating equilibria are

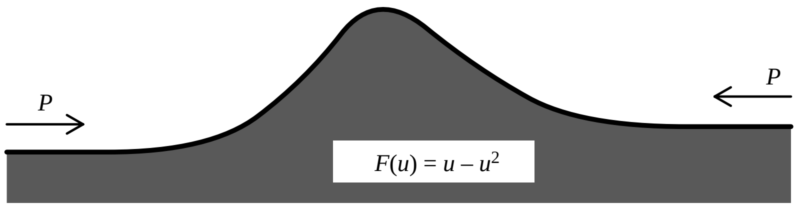

Figure 1. A strut on an asymmetric softening foundation with restoring force F under an axial load P.

multimodal solutions resembling concatenated, widely spaced copies of the primary state. There is numerical evidence that the primary buckling mode h persists up to $P = 2$ while still being transversely constructed (see Buffoni *et al.* 1996 and the references cited therein). The bifurcation to multimodal solutions mentioned above would then occur for all $P \in (-2, 2)$.

It is an interesting problem to determine whether these localized buckling modes are stable in an appropriate sense. In the present context, we adopt the notion of stability used in mechanical engineering and minimize the energy. For given axial load P, the total potential energy of localized equilibria is given by

$$W(u) = \frac{1}{2}\int_{-\infty}^{\infty}(u_{xx}^2 - Pu_x^2 + u^2 - \tfrac{2}{3}u^3)\,\mathrm{d}x, \tag{1.2}$$

(see, for example, Hunt *et al.* 1989). We consider two different loading devices for (1.1). Under *dead loading*, the deflection u adjusts according to (1.1). Accessing stability of an equilibrium under dead loading is then equivalent to minimizing the total energy $W(u)$ for fixed P (see Thompson & Hunt 1973). This task can be accomplished by verifying positive definiteness of the second variation $\nabla^2 W(h)$ of W given by

$$L(h)v := \nabla^2 W(h)v = v_{xxxx} + Pv_{xx} + (1-2h)v, \tag{1.3}$$

at a buckling state h. Note that, on account of translation invariance of $W(u)$, $L(h)$ has an eigenvalue at zero. Under *rigid loading*, the axial load P and the total displacement

$$\mathcal{E}(u) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2\,\mathrm{d}x, \tag{1.4}$$

are fixed. We then have to minimize $W(u)$ under the additional constraint

$$\mathcal{E}(u) = \text{const.},$$

(see Thompson 1979, ch. 9). This amounts to proving positive definiteness of $L(h)$ restricted to the kernel of $\nabla\mathcal{E}(h)$.

In this paper, stability of localized solutions h of (1.1) is addressed under both dead and rigid loading by analytical and numerical techniques. In § 2, stability of the primary buckling state is investigated. It is shown by analytical means that the primary mode is unstable under dead loading for all $P < 2$. For $P \in (-\infty, -2+\eta)$, we prove that there is precisely one negative eigenvalue. For struts under rigid loading, an integral condition for stability is derived analytically. Yet we are not able to verify this condition rigorously. However, using numerical techniques, it is likely that the primary state is unstable under rigid loading for $P < P_*$ and stable for $P > P_*$, where $P_* \approx 0.8175$. In § 3, we address the issue of stability of the multimodal states existing for $P \in (-2, 2)$. It is shown that all multimodal solutions consisting of widely spaced copies of the primary pulse are unstable under dead and rigid loading. Moreover, we determine n critical eigenvalues near zero of the second variation $\nabla^2 W$ evaluated at n-modal buckling states. They can be used to characterize obstacles for the coalescence of different multimodal states in saddle-node or pitchfork bifurcations. The results presented in § 3 hold for fairly general systems under generic assumptions on the nature of the primary mode. Thus, they may be applicable to equations for which the primary mode is stable guaranteeing existence of infinitely many stable multimodal states. In addition, existence of multimodal states is shown for $2m$-dimensional equations extending previous results obtained by Devaney (1976), Champneys (1994) and Härterich (1993). Finally, in § 4, we comment on other equations exhibiting similar phenomena.

2. Instability of the primary buckling mode

We shall determine the spectrum of the second variation

$$L(h)v = v_{xxxx} + Pv_{xx} + (1 - 2h)v, \tag{2.1}$$

of the total energy evaluated at a localized solution, h, of equation (1.1). The operator $L(h)$ is self-adjoint in $L^2(\mathbb{R})$ with domain $H^4(\mathbb{R})$. Therefore, its spectrum is confined to the real line. Throughout, we denote the L^2-scalar product by $\langle \cdot\,, \cdot \rangle$.

(a) *Dead loading*

In this section, we show that the primary buckling state is unstable under dead loading. As mentioned in § 1, this is tantamount to proving the existence of at least one negative eigenvalue of the second variation $L(h)$ of the total energy W. Indeed, the potential energy is then not minimized and any small imperfection will cause the mechanical system to snap into a different stable equilibrium.

Lemma 2.1. *Let $P < 2$ and h be a localized solution of (1.1).*
(i) The essential spectrum of $L(h)$ is then given by $\sigma_{\mathrm{ess}}(L(h)) = [1, \infty)$.
(ii) The operator $L(h)$ has at least one negative eigenvalue.
(iii) If the localized solution h is transversely constructed, zero is a simple eigenvalue of $L(h)$. In particular, when P is varied, the number of negative eigenvalues stays constant until h ceases to be transversely constructed.

Proof. Since $P < 2$, $h(x)$ converges to zero exponentially as $|x|$ tends to infinity. Therefore, the essential spectrum of $L(h)$ coincides with the interval $[1, \infty)$ (see Henry 1981, Appendix to § 5). We prove (ii) next. Multiplying (1.1) with h, integrating over x, and using $P < 2$, we have

$$\int_{-\infty}^{\infty} h^3 \,\mathrm{d}x = \int_{-\infty}^{\infty} (h_{xx}^2 - Ph_x^2 + h^2)\mathrm{d}x > \int_{-\infty}^{\infty} (h_{xx} + h)^2 \mathrm{d}x \geqslant 0.$$

Therefore,

$$\langle L(h)h, h\rangle = \int_{-\infty}^{\infty} (h_{xx}^2 - Ph_x^2 + h^2 - 2h^3)\mathrm{d}x = -\int_{-\infty}^{\infty} h^3 \,\mathrm{d}x < 0.$$

Thus, we conclude the existence of (at least) one negative eigenvalue of $L(h)$.

Any eigenvalue of $L(h)$ has geometric multiplicity less or equal to two. Since $L(h)$ is self-adjoint, all eigenvalues are semi-simple. If h is transversely constructed, it is clear that the eigenvalue zero with eigenfunction h_x is simple. Therefore, no eigenvalues

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can cross the imaginary axis upon changing P. On the other hand, since $L(h)$ is sectorial, no eigenvalues can escape to minus infinity either. ■

Next, we consider the transversely constructed primary buckling state which exists for $P < -2 + \eta$ by results of Buffoni *et al.* (1996).

Lemma 2.2. *Let $P \in (-\infty, -2+\eta)$ and consider the operator $L(h)$ evaluated at the transversely constructed primary buckling mode. There exists then precisely one negative eigenvalue λ_0 of $L(h)$.*

Proof. Due to lemma 2.1(ii), the operator has at least one negative eigenvalue. It remains to prove uniqueness. On account of transversality of the primary mode and lemma 2.1(iii), it suffices to prove the claim for $P \to -\infty$.

We exploit a coordinate transformation introduced by Amick & Toland (1992) for $P \to -\infty$. Let $-P = \sqrt{\epsilon} + (1/\sqrt{\epsilon})$ and $y = \epsilon^{1/4}x$. In the new coordinates, the localized solutions are denoted $h_\epsilon(y)$ making the dependence on the parameter ϵ explicit. By (Amick & Toland 1992), the continuous family h_ϵ converges in the sup-norm toward $h_0(y) := \frac{3}{2}\operatorname{sech}^2(\frac{1}{2}y)$ as $\epsilon \to 0$. The operator $L(h)$ transforms into

$$L_\epsilon v := \epsilon v_{yyyy} - (1+\epsilon)v_{yy} + (1 - 2h_\epsilon(y))v. \tag{2.2}$$

The eigenvalue problem for L_0 at $\epsilon = 0$ reads

$$-v_{yy} + (1 - 3\operatorname{sech}^2(\tfrac{1}{2}y))v = \lambda v, \quad v \in H^4(\mathbb{R}). \tag{2.3}$$

Changing coordinates according to $z = \tanh(\frac{1}{2}y)$ transforms (2.3) into the Legendre equation

$$(1-z^2)v_{zz} - 2zv_z + \left(12 - \frac{4(1-\lambda)}{1-z^2}\right)v = 0, \quad z \in [-1,1].$$

Applying Abramowitz & Stegun (1972, ch. 8), it is straightforward to calculate the eigenvalues λ_n of (2.3). We obtain $\lambda_0 = -\frac{5}{4}$, $\lambda_1 = 0$, and $\lambda_2 = \frac{3}{4}$. The eigenfunctions associated with λ_0 and λ_1 are $v_0(y) = \operatorname{sech}^3(\frac{1}{2}y)$ and $v_1(y) = (\mathrm{d}/\mathrm{d}y)h_0(y)$, respectively. Since L_0 is sectorial, there exists then a constant $K > 0$ such that

$$\langle L_0 v, v\rangle \geqslant K|v|^2_{H^1(\mathbb{R})}, \quad \text{for all } v \in (\operatorname{span}\{v_0, v_1\})^\perp \cap H^1(\mathbb{R}).$$

For positive $\epsilon > 0$ sufficiently small and all $v \in (\operatorname{span}\{v_0, v_1\})^\perp \cap H^2(\mathbb{R})$, we have therefore

$$\begin{aligned}\langle L_\epsilon v, v\rangle &= \int_{-\infty}^{\infty} (\epsilon v_{yy}^2 + (1+\epsilon)v_y^2 + (1-2h_\epsilon)v^2)\,\mathrm{d}y \\ &\geqslant \epsilon(|v_{yy}|^2_{L^2(\mathbb{R})} + |v_y|^2_{L^2(\mathbb{R})}) + K|v|^2_{H^1(\mathbb{R})} - 2|h_\epsilon - h_0|_{C^0(\mathbb{R})}|v|^2_{L^2(\mathbb{R})} \\ &\geqslant \tfrac{1}{2}K|v|^2_{L^2(\mathbb{R})},\end{aligned} \tag{2.4}$$

since $|h_{\epsilon_n} - h_0|_{C^0(\mathbb{R})} \to 0$ as $n \to \infty$. We will now argue by contradiction in order to prove uniqueness of the negative eigenvalue of the operator L_ϵ for $\epsilon > 0$. Thus, assume that there exists a sequence $\epsilon_n \searrow 0$ such that L_{ϵ_n} has two or more strictly negative eigenvalues. Note that zero is always a (simple) eigenvalue. The generalized eigenspace spanned by the associated eigenfunctions has therefore dimension greater or equal to three. In particular, its intersection with the space $(\operatorname{span}\{v_0, v_1\})^\perp$ is non-trivial. For any n, choose a function w_n with $|w_n|_{L^2(\mathbb{R})} = 1$ in this intersection. Note that $w_n \in H^2(\mathbb{R})$ since eigenfunctions are actually smooth. The inequality

$$\langle L_{\epsilon_n} w_n, w_n\rangle \leqslant 0$$

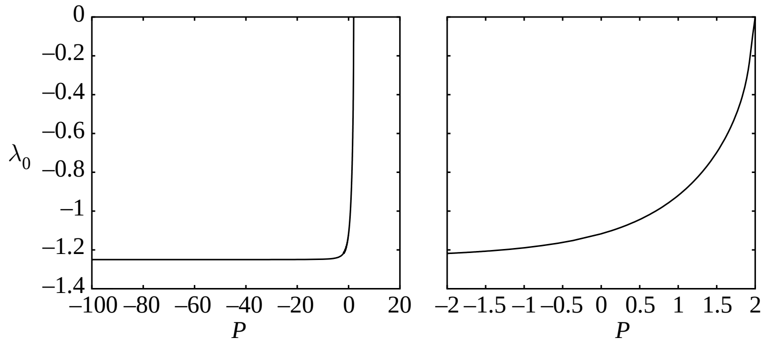

Figure 2. The negative eigenvalue λ_0 of the operator $L(h)$ at the primary buckling mode. For $P = -100$, we have $\lambda_0 = -1.24998 \approx -\frac{5}{4}$.

then holds. Substituting (2.4), we obtain

$$0 \geqslant \langle L_{\epsilon_n} w_n, w_n \rangle \geqslant \tfrac{1}{2} K |w_n|^2_{L^2(\mathbb{R})},$$

and reach a contradiction. This proves the lemma. ■

Combining lemmata 2.1(iii) and 2.2, we see that there exists a unique negative eigenvalue of $L(h)$ as long as h is transversely constructed.

We computed the negative eigenvalue λ_0 of $L(h)$ using the driver HOMCONT (see Champneys *et al.* 1995, 1996) for the software package AUTO written by Doedel & Kernévez (1986). Projection boundary conditions with respect to the constant-coefficient operator $v_{xxxx} + P v_{xx} + (1-\lambda) v$ are employed for the eigenfunction v_0. The initial guess $v_0(y) = \operatorname{sech}^3(\frac{1}{2}y)$, $\lambda_0 = -\frac{5}{4}$ at $P = -100$ has been used. Continuation in P shows that the negative eigenvalue persists up to $P = 2$ (see figure 2).

(*b*) *The dynamical problem*

Vibrations of the primary mode under dead loading are governed by the nonlinear equation:

$$u_{tt} + u_{xxxx} + P u_{xx} + u - u^2 = 0 \quad x \in \mathbb{R} \tag{2.5}$$

(see Lindberg & Florence 1987). We rewrite (2.5) as the first-order system

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} v \\ -u_{xxxx} - P u_{xx} - u + u^2 \end{pmatrix}. \tag{2.6}$$

The linearization of (2.6) at a buckling mode $(u, v) = (h, 0)$ is given by

$$\mathcal{L}(h, 0) = \begin{pmatrix} 0 & \mathrm{id} \\ -\partial_{xxxx} - P\partial_{xx} - \mathrm{id} + 2h & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -L(h) & 0 \end{pmatrix}.$$

Equation (2.6) generates a C^0-semiflow on the space $H^2(\mathbb{R}) \times L^2(\mathbb{R})$ (see, for example, Pazy 1983). For $P < 2$, $L(h)$ has at least one negative eigenvalue λ_0 (see lemma 2.1). Thus, the operator L has two eigenvalues $\pm\sqrt{-\lambda_0}$ on the real axis. The following lemma is then a consequence of Grillakis (1988, Appendix).

Lemma 2.3. *Any buckling mode of (1.1) under dead loading is unstable with respect to equation (2.5). In other words, there are a neighbourhood U of $(h, 0)$ and initial conditions arbitrary close to $(h, 0)$ such that the associated solutions leave the set U.*

The conclusion of lemma 2.3 is also true for parabolic or damped hyperbolic versions of equation (2.5).

(c) Rigid loading

Rigid loading is characterized by introducing an additional constraint: only imperfections with a prescribed total deflection are admissible. We therefore consider the system

$$u_{xxxx} + Pu_{xx} + u - u^2 = 0, \quad \mathcal{E}(u) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2 \, \mathrm{d}x = \text{const.}$$

Stability of the buckling mode h is equivalent to proving positive definiteness of the second variation $L(h)$ defined in (1.3) restricted to the kernel of the gradient $\nabla\mathcal{E}(h)$ of the constraint $\mathcal{E}$. The gradient $\nabla\mathcal{E}(h) : L^2(\mathbb{R}) \to \mathbb{R}$ is given by

$$\nabla\mathcal{E}(h)v = \langle h_{xx}, v\rangle = \int_{-\infty}^{\infty} h_{xx} v \, \mathrm{d}x.$$

First, we exploit that the functional $\mathcal{E}(u)$ is invariant under translation. As a consequence, $h_x \in \ker \nabla\mathcal{E}(h)$ since $\int_{-\infty}^{\infty} h_{xx}h_x \, \mathrm{d}x = 0$ by integration by parts. We may therefore restrict $L(h)$ to the space

$$H := \ker \nabla\mathcal{E}(h) \cap (\operatorname{span}\{h_x\})^{\perp} \subset L^2(\mathbb{R}).$$

Note that H is of codimension two. We have then to verify that

$$\langle L(h)v, v\rangle > 0, \quad v \in H. \tag{2.7}$$

Remark 2.4. *A buckling state h can only be stable under rigid loading if h is transversely constructed and the operator $L(h)$ has a unique negative eigenvalue.*

Proof. By lemma 2.1, $L(h)$ has at least two eigenvalues less or equal to zero. Assume that there are at least three of them. There is then a three-dimensional subspace W of $L^2(\mathbb{R})$ such that $h_x \in W$ and $\langle L(h)w, w\rangle \leqslant 0$ for all $w \in W$. Therefore, $W \cap H$ is at least one dimensional, and (2.7) is not satisfied. ■

It remains therefore to consider transversely constructed localized solutions h. We remark that whenever a localized solution h is transversely constructed at $P = P_0$ there exists a unique family $h(P)$ of buckling modes defined for P close to P_0 such that $h(P_0) = h$. With this remark in mind, we formulate the next lemma.

Lemma 2.5. *Let $P < 2$. Assume that h is transversely constructed such that $L(h)$ has precisely one negative eigenvalue. Condition (2.7) is then met if, and only if,*

$$\frac{\mathrm{d}}{\mathrm{d}P}\mathcal{E}(h(P)) < 0. \tag{2.8}$$

Here, $h(P)$ denotes the family of localized solutions mentioned before the lemma.

In other words, $h(P)$ is stable at $P = P_0$ under rigid loading if, and only if, $\mathcal{E}(h(P))$ decreases strictly near P_0.

Proof. Since zero is a simple eigenvalue of $L(h)$ by assumption, the operator $L(h)$ is invertible on $(\operatorname{span}\{h_x\})^{\perp}$. It is a consequence of Alexander *et al.* (1996), proof of lemma 2 (see also Maddocks (1985)), that $L(h)$ is positive on H if, and only if,

$$\langle L(h)^{-1}h_{xx}, h_{xx}\rangle < 0. \tag{2.9}$$

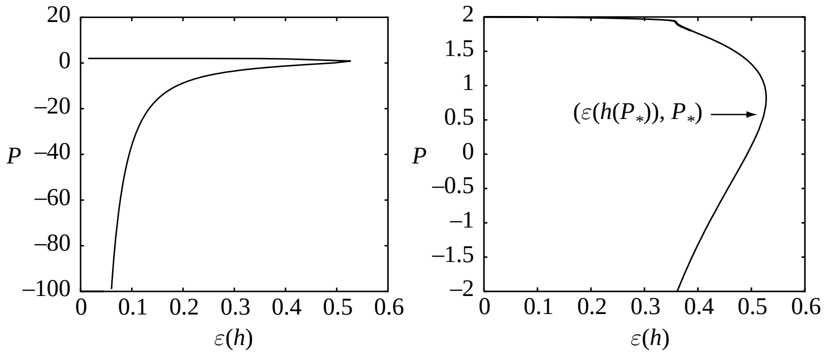

Figure 3. The integral constraint $(\mathcal{E}(h(P)),P)$. The function $\mathcal{E}(h(P))$ is increasing for $P<P_*$ and decreasing for $P>P_*$. At $P=P_*\approx 0.8175$ an exchange of stability occurs. There is an anomaly on the top of the right-hand curve which can also be found in Wadee *et al.* (1996, figure 6).

On the other hand, $h(P)$ satisfies (1.1) for all P close to P_0. Differentiating (1.1) with respect to P, we see that $(\mathrm{d}/\mathrm{d}P)h(P_0)$ satisfies $L(h)(\mathrm{d}/\mathrm{d}P)h(P_0)=-h_{xx}$. Hence, $L(h)^{-1}h_{xx}=-(\mathrm{d}/\mathrm{d}P)h(P_0)$, and (2.9) reads

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}P}h(P_0),h_{xx}\right\rangle > 0.$$

Differentiating $\mathcal{E}(h(P))=\int_{-\infty}^{\infty}h_x^2(P)\,\mathrm{d}x$ with respect to P and integrating by parts, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}P}\mathcal{E}(h(P))=-\left\langle \frac{\mathrm{d}}{\mathrm{d}P}h(P_0),h_{xx}\right\rangle < 0.$$

Thus, (2.7) and (2.8) are equivalent. ∎

We were not able to verify condition (2.8) rigorously. Numerical simulations using HOMCONT suggest that the primary buckling state is stable under rigid loading for $P\in(P_*,2)$, while it is unstable for $P\in(-\infty,P_*)$ (see figure 3). Here, $P_*\approx 0.8175$.

3. Stability of multimodal solutions

Consider equation (1.1) and the eigenvalue problem for (1.3)

$$u_{xxxx}+Pu_{xx}+u-u^2=0, \tag{3.1}$$

$$L(u)v=v_{xxxx}+Pv_{xx}+(1-2u)v=\lambda v. \tag{3.2}$$

We assume that a transversely constructed localized solution h_1 of (3.1) has been found. Then, localized *multimodal* states h_n are sought which resemble n copies of h_1 widely spaced in x. Alexander *et al.* (1990) proved that $L(h_n)$ has n eigenvalues counted with multiplicity near any eigenvalue of $L(h_1)$. In particular, invoking lemma 2.1(iii), there are precisely n eigenvalues of $L(h_n)$ close to zero. In § 3 *a*, we will calculate these eigenvalues. To achieve this, we rewrite (3.1) and (3.2) as first-order systems

$$U_x=f(U), \tag{3.3}$$

$$V_x=(Df(U)+\lambda B)V, \tag{3.4}$$

where $U, V \in \mathbb{R}^4$, $f(U) := (U_2, U_3, U_4, -PU_3 - U_1 + U_1^2)$ and $BV := (0,0,0,V_1)$. Section 3 *b* contains stability results for the strut equation under both dead and rigid loading as well as some remarks on exclusion principles for coalescence of multimodal states.

(*a*) *The linearized eigenvalue problem*

We will consider

$$u_x = f(u), \tag{3.5}$$

$$v_x = (Df(u) + \lambda B)v, \tag{3.6}$$

for $u, v \in \mathbb{R}^{2m}$ and $\lambda \in \mathbb{C}$. Here, $B \in \mathbb{R}^{2m\times 2m}$, and $f : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is smooth with $f(0) = 0$.

Hypothesis 3.1. *Assume that the spectrum $\sigma(Df(0))$ contains simple eigenvalues $\pm\alpha \pm \mathrm{i}\beta$ for some $\alpha, \beta > 0$. The modulus of the real part of any other eigenvalue of $Df(0)$ is strictly larger than α.*

We assume that q_1 is a transversely constructed localized solution of (3.5). As a consequence, the equation

$$w_x = -Df(q_1)^* w, \tag{3.7}$$

has, up to constant multiples, a unique bounded solution $\psi(x)$ (see Sandstede 1997). In fact,

$$\psi(x) \perp (T_{q_1(x)}W^s(0) + T_{q_1(x)}W^u(0)).$$

Suppose now that at least one of the following two hypotheses is satisfied:

Hypothesis 3.2. (Reversible systems)**.**
(i) Suppose that $R : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is a linear involution with $\dim \operatorname{Fix} R = m$ such that $f(Ru) = -Rf(u)$ for all u. Furthermore, assume that q_1 is symmetric, i.e. $q_1(x) = -Rq_1(-x)$.
(ii) Assume that

$$\lim_{x\to\infty} \mathrm{e}^{2\alpha x}|q_1(x)||\psi(x)| > 0.$$

or

Hypothesis 3.3. (Conservative systems)**.**
(i) Suppose that $H : \mathbb{R}^{2m} \to \mathbb{R}$ is a smooth function such that $\nabla H(q_1(0)) \neq 0$ and $\langle f(u), \nabla H(u)\rangle = 0$ for all u.
(ii) In addition, suppose that

$$\lim_{x\to\infty} \mathrm{e}^{2\alpha x}|q_1(x)||q_1(-x)| > 0.$$

Hypotheses 3.2 and 3.3 are satisfied for generic reversible or conservative systems. Note that any Hamiltonian system is in particular conservative as energy is preserved. In this case, the solution $\psi(x)$ of (3.7) is readily computed.

Remark 3.4. *Suppose that hypothesis 3.3 is satisfied. It is then straightforward to check that $\psi(x) = \nabla H(q_1(x))$. In particular, we obtain the asymptotic expansion $\psi(x) = \nabla^2 H(0)q_1(x) + O(|q_1(x)|^2)$.*

Finally, we assume a Melnikov-type condition.

Hypothesis 3.5. $M := \int_{-\infty}^{\infty} \langle \psi(x), B(\mathrm{d}/\mathrm{d}x)q_1(x)\rangle \, \mathrm{d}x \neq 0.$

We then define the sets

$$\mathcal{R} = \left\{ \exp\left(-\frac{2\pi\alpha}{\beta} n \right); n \in \mathbb{N}_0 \right\} \cup \{0\}, \quad \mathcal{A} = \left\{ \exp\left(-\frac{\pi\alpha}{\beta} k \right); k \in \mathbb{N}_0 \right\}. \tag{3.8}$$

Note that $\mathcal{R}$ is a closed metric space. These sets will be used to characterize multimodal solutions. We need one more definition. Suppose that hypothesis 3.1 is satisfied. A sequence $(x_j)_{j=1,\ldots,k}$ is then called *admissible* if, and only if, $x_{k+1-j} = x_j$. In case hypothesis 3.2 holds, *any* sequence is called admissible. The main theorem can now be stated.

Theorem 3.6. *Let q_1 be a transversely constructed localized solution of (3.1). Assume that hypotheses 3.1 and 3.5, and, in addition, 3.2 or 3.3 (or both) are satisfied. There exists then a $\delta > 0$ such that for any $n \geqslant 2$ the following holds.*

For any admissible sequence $a_j^0 \in \mathcal{A}$ with $j = 1, \ldots, n-1$ and $a_i^0 \in \{1, \exp(-\pi\alpha/\beta)\}$ for some i, there exists an $r_0 \in \mathcal{R}$, $r_0 \neq 0$, with the following property.

(i) There are C^0-functions $a_j(r) \in \mathbb{R}$ for $r \in \mathcal{R}$, $r \leqslant r_0$, with $a_j(0) = a_j^0$ for $j = 1, \ldots, n-1$.

(ii) For any $r \in \mathcal{R}$ with $0 < r \leqslant r_0$, there exists an n-modal solution q_n. The distances between consecutive copies of q_1 in the n-modal solution q_n are given by

$$L_j(r) = -\frac{1}{\alpha} \ln(a_j(r) r) + \tilde{L}, \quad j = 1, \ldots, n-1,$$

for some constant $\tilde{L}$.

(iii) The n-modal orbits satisfying (ii) are unique. If hypothesis 3.2 holds, they are in addition symmetric.

Denote the natural numbers associated with the $a_j^0 \in \mathcal{A}$ chosen above by k_j^0. There are then precisely n solutions λ_j of (3.6) evaluated at q_n inside a ball of radius δ around zero, and

(iv) for $M > 0$ ($M < 0$), we have

$$\#\{j; 1 \leqslant j \leqslant n-1, \operatorname{Re}\lambda_j < 0\} = \#\{j; 1 \leqslant j \leqslant n-1, k_j^0 \text{ is odd (even)}\}$$
$$\#\{j; 1 \leqslant j \leqslant n-1, \operatorname{Re}\lambda_j > 0\} = \#\{j; 1 \leqslant j \leqslant n-1, k_j^0 \text{ is even (odd)}\}$$

counted with multiplicity. Moreover, $\lambda_n = 0$ is a simple eigenvalue.

Note that the natural numbers $k_j^0 + n^0(r)$ associated with each a_j^0 and r, and hence with any n-modal state described in theorem 3.6, can be interpreted as the number of half-twists the n-modal state makes near the zero equilibrium.

Existence results of multimodal solutions in four dimensions have been first proved by Devaney (1976), Champneys (1994) and Härterich (1993). There are no rigorous stability results for multimodal solutions. Buryak & Akhmediev (1995) argued formally for a coupled nonlinear Schrödinger equation that multimodal solutions should be unstable; however, they used a criterion which is only necessary but not sufficient for instability.

Applying theorems 1 and 3 of Sandstede (1995) provides another way of extending the existence results from four- to higher-dimensional systems. In addition, Sandstede (1995, theorem 1) shows that the recurrent dynamics near the primary mode q_1 is confined to a four-dimensional locally invariant and normally hyperbolic 'centre' manifold W^{c}_{hom} containing q_1.

Before we prove theorem 3.6 in § 3 *c*, some consequences for the strut model are presented.

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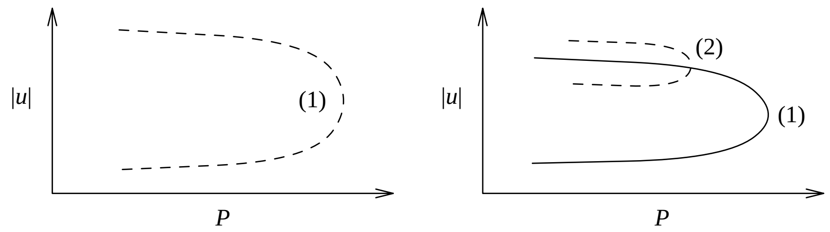

Figure 4. Coalescence of symmetric (solid) and asymmetric (dashed) multimodal solutions via saddle-node (1) and pitchfork (2) bifurcations (see Buffoni *et al.* 1996, figure 24).

(*b*) *Consequences for the strut model*

We have the following instability result for multimodal states.

Lemma 3.7. *Suppose that h is a localized solution of (1.1). Any multimodal state composed of copies of h is then unstable under dead and rigid loading.*

Proof. Note that $L(h)$ has at least one negative eigenvalue due to lemma 2.1(ii). We denote the multimodal state by h_n and assume that consecutive humps are widely separated. It follows then from Alexander *et al.* (1990) that $L(h_n)$ has at least n negative eigenvalues. Therefore, any multimodal state is unstable under dead loading. That it is also unstable under rigid loading follows from remark 2.4. ■

For $P \in (-2, -2+\eta)$ with $\eta > 0$ sufficiently small, Buffoni *et al.* (1996) proved that the primary buckling mode is transversely constructed and that hypothesis 3.1 is met. Hypotheses 3.2 and 3.3 are also satisfied. Note that hypothesis 3.2(ii) and 3.3(ii) are automatically met since (1.1) is four dimensional. Finally, hypothesis 3.5 is true. Indeed, using remark 3.4, and the definitions $q = (h, h_x, h_{xx}, h_{xxx})$ and $Bq_x = (0, 0, 0, h_x)$, we obtain

$$M = \int_{-\infty}^{\infty} \langle \psi, Bq_x \rangle \mathrm{d}x = \int_{-\infty}^{\infty} \langle \nabla H(h(x)), (0,0,0,h_x(x)) \rangle \mathrm{d}x = \int_{-\infty}^{\infty} h_x^2 \, \mathrm{d}x > 0.$$

Therefore, theorem 3.6 applies to (1.1). Buffoni *et al.* (1996) observed numerically the coalescence of symmetric and asymmetric multimodal buckling states (see figure 4). Generically, coalescence is expected to occur via saddle-node or pitchfork bifurcations in the underlying partial differential equation.

Such bifurcations are related to an exchange of stability of the contributing n-modal localized solutions. In particular, generically, precisely one eigenvalue should cross the imaginary axis at zero; remember that the operators involved are self-adjoint. Therefore, the indices, i.e. the number of negative eigenvalues, of the involved n-modal solutions should differ by one. Note that we have to take the n negative eigenvalues near $\lambda_0 < 0$ into account. That immediately prevents coalescence of n_1-modal and n_2-modal solutions whenever $n_1 > 2n_2$. Furthermore, observe that the sequence $(k_j^0)_{j=1,\ldots,n-1}$ associated with each n-modal symmetric state is admissible, i.e. $k_j^0 = k_{n-j}^0$ holds for all j. In particular, eigenvalues of n-modal symmetric states come in pairs except for the eigenvalue zero and possibly another eigenvalue if n is even. Thus, it is likely that just before two symmetric states coalesce one eigenvalue has to cross the imaginary axis in a pitchfork bifurcation. No such obstacle exists for asymmetric states. A more refined criterion than just counting unstable eigenvalues is to include the symmetry of the corresponding eigenfunctions. Indeed, both, the number of even and odd unstable eigenfunctions, have to coincide for coalescence of

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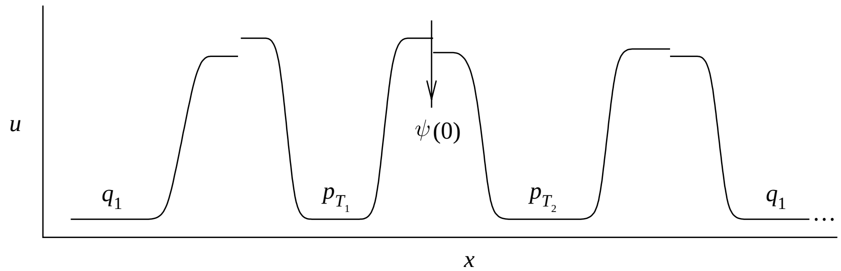

Figure 5. The discontinuous solution for $N = 3$ with distances T_1 and T_2. The discontinuities are contained in the line spanned by $\psi(0)$.

two multimodal states to occur. For each particular sequence (k_j^0), these numbers can be calculated by computing the eigenvectors of the matrix A_0 appearing in the proof of theorem 3.6.

(*c*) *The proof of theorem 3.6*

We employ homoclinic Ljapunov–Schmidt reduction as developed in Lin (1990) and Sandstede (1993) for existence, and in Sandstede (1997) for stability. Since this method is rather technical, we shall illustrate it for two-dimensional systems, i.e. $u \in \mathbb{R}^2$, and refer to the articles mentioned above for the details in several dimensions.

Consider therefore a two-dimensional conservative system, say, and assume that q_1 is a homoclinic solution. It is then accompanied by a family p_T of periodic solutions parametrized by their period $2T$. We may assume that $p_T(0) = q_1(0) + \delta\nabla H(q(0))$ for some small δ depending on p_T. Then p_T converges to h uniformly on the interval $(-T, T)$ as $T \to \infty$. We seek multimodal solutions (which, of course, do not exist in two dimensions). The idea is then to use the distances between consecutive copies of q_1 as variables. Therefore, choose a sequence T_j for $j = 1, \dots, n-1$ with $T_j > 0$ large such that $2T_j$ is a candidate for the distance between the jth and the $(j+1)$th copy of q_1 in the n-modal solution we want to construct. We may then concatenate the homoclinic orbit q_1 restricted to $\mathbb{R}^-$ with the solutions p_{T_j} defined on $(-T_j, T_j)$ for $j = 1, \dots, n-1$, and finally with the other tail of q_1 defined on $\mathbb{R}^+$ (see figure 5 for $n = 3$). The resulting function looks like an n-modal solution; it is, however, discontinuous whenever it hits the line

$$\{q_1(0) + \delta\nabla H(q_1(0)); \delta \in \mathbb{R}\} = \{q_1(0) + \delta\psi(0); \delta \in \mathbb{R}\}.$$

Therefore, in order to show existence of multimodal orbits, it suffices to prove existence of numbers T_j such that all these discontinuities disappear. Since the discontinuous solution meets the line n times, there are n equations which have to be solved. If we seek symmetric solutions, we only need to solve the first $\frac{1}{2}n$ equations corresponding to the interval $(-\infty, 0]$. Indeed, we may then extend the solution to $(-\infty, \infty)$ using the symmetry. Similarly, for conservative systems, the last equation is always satisfied provided the first $n-1$ equations are: the remaining discontinuity occurs in the direction of increasing or decreasing energy, and is therefore zero since both tails have the same energy by construction.

A similar technique works in higher dimensions. As a result, we obtain a solution with n discontinuities which are contained in the line spanned by $\psi(0)$. If the discontinuities disappear, a multimodal solution is constructed. It is then possible, though a non-trivial exercise, to compute these discontinuities. Indeed, they are given by

$$\langle\psi(T_j), q_1(-T_j)\rangle - \langle\psi(-T_{j-1}), q_1(T_{j-1})\rangle - R_j((T_i)_{i=1,\dots,n-1}) = 0, \qquad (3.9)$$

for $j = 1, \ldots, n$ and large T_j (see Sandstede 1993, Satz 3). The remainder terms R_j are of higher order. Existence of n-modal solutions is equivalent to solving (3.9). We then have the following proposition.

Proposition 3.8. *Fix $n \geqslant 2$. There exists an n-modal homoclinic orbit q_n of (3.5) close to q_1 in phase space if, and only if,*

$$\left.\begin{aligned} a_1 \sin(-(\beta/\alpha)\ln(a_1 r)) &= R_1(a,r), \\ a_{j-1}\sin(-(\beta/\alpha)\ln(a_{j-1}r)) - a_j \sin(-(\beta/\alpha)\ln(a_j r)) &= R_j(a,r), \\ a_{n-1}\sin(-(\beta/\alpha)\ln(a_{n-1}r)) &= R_n(a,r), \end{aligned}\right\} \tag{3.10}$$

is satisfied where $j = 2, \ldots, n-1$. In fact, if hypothesis 3.2 holds, it suffices to solve (3.10) for $j = 1, \ldots, [\frac{1}{2}n]$, where $[x]$ denotes the largest integer smaller than x; if hypothesis 3.3 is satisfied, it is sufficient to solve (3.10) for $j = 1, \ldots, n-1$. The remainder terms $R_j(a,r)$ are smooth in $a = (a_j)$ for $a_j \in (0,1]$ up to $r = 0$ and

$$R_j(a,r) = O(r^\gamma), \quad \frac{\mathrm{d}}{\mathrm{d}a_i} R_j(a,r) = O(r^\gamma), \tag{3.11}$$

for some $\gamma > 0$.

Proof. As mentioned above, Sandstede (1993, Satz 3) implies that existence of n-modal states is equivalent to solving (3.9). In order to derive (3.10) from (3.9), we proceed as in Sandstede (1997, § 6) and refer to that article for the details. The statement about the number of equations which need to be solved is proved in Sandstede *et al.* (1997, lemmata 3.1 and 3.2), respectively. In addition, we exploit the following symmetries. In the conservative case, hypothesis 3.3, we have

$$\begin{aligned} \langle \psi(x), q_1(-x)\rangle &= \langle \nabla^2 H(0) q_1(x), q_1(-x)\rangle + O(|q_1(x)|^2 |q_1(-x)|) \\ &= \langle q_1(x), \nabla^2 H(0) q_1(-x)\rangle + O(|q_1(x)|^2 |q_1(-x)|) \\ &= \langle q_1(x), \psi(-x)\rangle + O(|q_1(x)||q_1(-x)|(|q_1(x)| + |q_1(-x)|)) \end{aligned}$$

by remark 3.4. Under hypothesis 3.2, we have $\langle \psi(x), q_1(-x)\rangle = \langle \psi(-x), q_1(x)\rangle$ using Sandstede (1997, lemma 5.3). ■

We consider the conservative case first and comment later on the changes necessary for the reversible case. So, assume that hypothesis 3.3 is met. The proof is similar to Sandstede (1997, proof of theorem 3), where saddle-focus bifurcations for generic, non-reversible systems have been studied.

By proposition 3.8, we have to solve

$$\begin{aligned} a_1 \sin(-(\beta/\alpha)\ln(a_1 r)) &= R_1(a,r) \\ a_{j-1}\sin(-(\beta/\alpha)\ln(a_{j-1}r)) - a_j \sin(-(\beta/\alpha)\ln(a_j r)) &= R_j(a,r), \end{aligned}$$

for $j = 2, \ldots, n-1$. Inserting $r = \exp(-(2\pi\alpha/\beta)n)$ with $n \in \mathbb{N}$, we obtain

$$\left.\begin{aligned} a_1 \sin(-(\beta/\alpha)\ln a_1) &= R_1(a,r), \\ a_{j-1}\sin(-(\beta/\alpha)\ln a_{j-1}) - a_j \sin(-(\beta/\alpha)\ln a_j) &= R_j(a,r), \end{aligned}\right\} \tag{3.12}$$

for $j = 2, \ldots, n-1$. This allows us to take the limit $r \to 0$ yielding

$$\begin{aligned} a_1 \sin(-(\beta/\alpha)\ln a_1) &= 0, \\ a_{j-1}\sin(-(\beta/\alpha)\ln a_{j-1}) - a_j \sin(-(\beta/\alpha)\ln a_j) &= 0, \end{aligned}$$

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for $j = 2, \dots, n-1$. However, these equations are satisfied by the chosen sequence $a^0 = (a_j^0)$ with $a_j^0 \in \mathcal{A}$. Moreover, the Jacobian of (3.12) with respect to a at $(a_j, r) = (a_j^0, 0)$ is lower triangular and the entries on the diagonal are given by

$$\left(\sin\left(-\frac{\beta}{\alpha} \ln a_j \right) - \frac{\beta}{\alpha} \cos\left(-\frac{\beta}{\alpha} \ln a_j \right) \right) \Big|_{a_j = a_j^0 = \exp(-(\pi\alpha/\beta)k_j^0)} = (-1)^{k_j^0+1} \left(\frac{\beta}{\alpha} \right), \tag{3.13}$$

and thus are non-zero. Therefore, since the remainder terms are differentiable due to proposition 3.8, an application of the implicit function theorem proves (i)–(iii) of the theorem in the conservative case.

It remains to prove (iv) which addresses the stability properties of the n-modal orbits. The general approach is very similar to the existence part. In fact, any eigenfunction is an n-modal solution with respect to the primary mode $(\mathrm{d}/\mathrm{d}x)q_1$ of the linear non-autonomous equation, (3.6). In Sandstede (1997), a general procedure has been developed which reduces the calculation of the critical eigenvalues to the computation of eigenvalues of an $n \times n$ matrix. Let q_n be an n-modal solution given by $(a(r), r)$. Then, proceeding as in Sandstede (1997, § 6), all eigenvalues $\lambda(r)$ close to zero of (3.6) evaluated at q_n are given by $\lambda(r) = r\nu(r)$ for some continuous function $\nu(r)$ such that $\nu(0)$ is an eigenvalue of the matrix A_0 given by

$$(A_0)_{ij} = \begin{cases} b_j + b_{j-1}, & j = i, \\ -b_{j-1}, & j = i-1, \\ -b_j, & j = i+1, \\ 0, & \text{otherwise}, \end{cases}$$

with $b_j = (-1)^{k_j^0} c \operatorname{sgn}(M)$ for some positive constant $c > 0$ and $j = 1, \dots, n-1$; we set $b_0 = 0$. Note that A_0 is tridiagonal, symmetric and the sum of its entries in each row vanishes. The number of positive and negative eigenvalues for such matrices has been determined in Sandstede (1997, lemma 5.4). Hence, statement (iv) of theorem 3.6 is proved for conservative systems.

In case (3.5) is reversible, we proceed as above. The only difference is that we solve (3.12) for $j = 1, \dots, [\frac{1}{2}n]$ instead for $j = 1, \dots, n-1$. This completes the proof of the theorem.

4. Discussion

There are several other fourth-order equations for which multimodal solutions do exist. Consider, for instance,

$$u_{tt} = -(u_{xxxx} + \mathrm{e}^u - 1), \tag{4.1}$$

$$u_t = -(\epsilon^2 u_{xxxx} - u_{xx} - u + u^3). \tag{4.2}$$

Equation (4.1) is known as the suspended beam model (see McKenna & Chen 1997), while (4.2), the extended Fisher–Kolmogorov equation, arises in the study of so-called Lifschitz points in phase transitions. Both equations exhibit similar features to (2.5) and therefore the results of §§ 2 and 3 are likely to apply to them as well. For instance, Peletier & Troy (1995*a*, *b*) proved existence of kinks for equation (4.2). Since these kinks bifurcate from the stable kinks of the Nagumo equation at $\epsilon = 0$, they are presumably also stable. Employing theorem 3.6 gives the existence of infinitely many stable multimodal kinks of (4.2) once its hypotheses are met. Note that these assumptions are generic within the class of reversible Hamiltonian equations.

Recently, Lord *et al.* (this volume) investigated the von Kármán–Donnell equations

$$\kappa^2\nabla^4 w + \lambda w_{xx} - \rho\phi = w_{xx}\phi_{yy} + w_{yy}\phi_{xx} - 2w_{xy}\phi_{xy},$$
$$\nabla^4\phi + \rho w_{xx} = (w_{xy})^2 - w_{xx}w_{yy},$$

with $(x, y) \in \mathbb{R} \times S^1$. They discretized the elliptic system in y and obtained a large system of ordinary differential equations in the unbounded variable x. Numerically, they found then a primary localized solution with oscillating tails. At least on the discretized level, theorem 3.6 can be used to confirm existence of multimodal solutions. In work in progress, Peterhof *et al.* (1997) currently extend theorem 3.6 to elliptic equations on unbounded domains and justify the numerical techniques used in Lord *et al.* (this volume).

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